Introduction

In 1859 the great German mathematician, Bernhard Riemann, presented to the Academy of Sciences in Berlin a paper titled *"Über die Anzahl der Primzahlen under einer gegebenen Grösse"* (On the Number of Prime Numbers Below A Given Quantity). This work was originally written in a manuscript of only six pages and yet it unfolds fascinating depth and originality. It has led to one of the most captivating mathematical problems in history: the Riemann Hypothesis.

Bernhard Riemann conjectured a hypothesis that generated the greatest mathematical movement to prove it. In an elegant and brilliant way Riemann's works connected the theory of prime numbers with mathematical analysis, opening multiple paths to later researchers, who, in turn, have led to surprising results. But, to this day (March 2020), the Riemann hypothesis has still not been proven, and continues to represent the greatest challenge even to the most talented mathematicians.

This book takes us along the wonderful historical journey necessary to understand Riemann's works, helping us to formulate his hypothesis and to study its most important implications.

Perhaps the greatest known mathematical secret is hidden in the Riemann Hypothesis. If the hypothesis were true, then we should look at prime numbers in a certain way. But, if it were false, then we should look at them from a very different point of view. We do not know.

I decided to write this book, which deals with the amazing world of prime numbers, their properties and history, after reading in recent years a large amount of literature on them. Wonderful literature created from the great masters of mathematics throughout history: Euclid, Euler, Gauss, Dirichlet, Riemann, Hadamard, Hardy, Littlewood, and many, many others that will arise throughout these pages.

But, in addition to having read about this, I took the determination due to the enormous attraction that I have felt since I was a child to the world of prime

numbers, in particular, and to mathematics in general. I am no mathematician: I am an engineer who likes mathematics. This attraction makes me write contributing with my own thoughts and, in particular, it makes me write about the thoughts of the great mathematicians. Because, let's say it right now, the prime numbers are not fully understood. No mathematician who has contributed to the advance in the knowledge of prime numbers, none, is entirely satisfied with their contributions. All, without exception, have experienced, at the end of their work, the slightly bitter feeling of not having reached the end, of having fallen by the wayside.

This does not mean that they have not made extraordinary advances, which will be discussed throughout this book. Totally unexpected amazing advances featuring people of high intelligence, that make us see more light, but not all the light, between the shadows that exist in this infinite set of numbers that we call primes.

It is amazing that something that is so simple to express, that something so primary in the wide world of mathematics, has led to such profound studies, to so much literature, to so many theorems, which are among the most complicated and yet subtle. Something as clear as prime numbers, which are but natural numbers with a particular property, which makes them unique, has troubled the minds of the greatest geniuses of history, not having reached, to this day, to fully understand their nature. It is that aura of mystery, this challenge to the intelligence that appears when we face them, the reason why so many people feel attracted to them. Me, among them, and very intensively.

Ultimately, writing a book about prime numbers can be a symptom of frustration. It can be said that, although we do not understand them completely, and no progress has been made in the knowledge of their ultimate nature, but at least we are moving forward in everything that surrounds them: their history, the knowledge of their properties, their relationship with other subjects and derivations that have resulted in disciplines other than mathematics.

Throughout history there have been excellent mathematicians partly frustrated by not achieving the goals they had set themselves, that we will learn throughout the chapters. And, also, equally excellent and wiser mathematicians, who have refused to address this matter by fear of being challenged with an impossible peak

to climb, ending up humiliated after such a strenuous effort. So many have failed before that any mathematician, before tackling a new way of climbing, should think about it carefully, lest the effort consume their energies during years of work, with the high risk of not reaching anything new.

The latter leads to a question: is this a complicated book? Not if the reader does not want to. This book is not for professional mathematicians, but for math students or amateurs. Prime numbers are inherent to us, so, in principle, understanding them should not pose any problem. All mathematical theories around them may be more or less profound, but not all of them are inaccessible to the average reader, if approached from the proper viewpoint. A scientific diploma level or first-year technical university degree are sufficient to understand this book in its entirety.

To prepare the reading of some chapters I have devoted to you, readers, introductions and presentations of mathematical concepts and tools that will be needed to better understand the content of the subject. What readers should not do is browse the book quickly or skip to main or end chapters with the desire to advance. This will be frustrating, because you will not understand many things, and equations will fall on you as heavy stones. An understood equation is a treasure. An equation that is not understood is a discouraging burden. Therefore, it is best to go chapter by chapter, without taking shortcuts. At the end of each chapter, for the more curious readers, I have included some pages with explanations and insights on mentioned points, and with the references made to other authors.

Prime numbers themselves are mysterious and attractive. But the history of mathematics that we know, from Euclid in the third century BC, until today, is also attractive. No less than twenty great and countless average mathematicians have devoted part of their lives to understand, and above all, to try to explain, what pattern primes have between natural numbers. But if I had to highlight four, I would say Euclid, Euler, Gauss and Riemann. They are the real pillars on which all other mathematicians have supported their work, who, although being excellent, do not reach the level of genius of the first.

Throughout the chapters we will discover the mathematical atmosphere of large universities and historical and current research centers, where the plethora of men

and women captivated by the primes worked. We will move through countries and continents: Greece initially, Europe until mid-twentieth century and America to this day. We will discover his work, as it was interpreted at the time, the schools they created and the disciples they had. Therefore, this book is not purely mathematical, it is also historical.

History is full of changes, wars and cultures. We must understand the historical context in which our characters lived between the eighteenth and twenty-first centuries. Mathematicians, who were normal people yet focused on their research, lived circumstances that affected their work. Knowing the history of mathematics is knowing the history of countries. Countries with renowned mathematicians have been and are today stronger countries. And vice versa. Therefore, the study and passion for mathematics are key to societies. Mathematics is not a subject to give short-term successes. Its achievements are counted by units of years, decades or centuries. Being a mathematician means being patient. Societies that strive to accomplish short-term achievements have no room for mathematics. This should be very clear.

Finally, I have ventured to write this book because, basically, I very much enjoy writing about a topic that I find very attractive and deep. Prime numbers is a subject that has gone through mathematical research, from the beginning to the end for over twenty three centuries, and that is not yet well known today. Primes bring together the simplest concepts, such as the fact of counting numbers, with the deeper aspects of mathematical analysis. Therein are hidden rules that remain mysterious. The greatest geniuses have dealt with them. However, being apparently so simple, they strongly attract those who are initiated in their knowledge and nature, as they discover the aspects of an unknown and surprising polyhedron. When approaching prime numbers, we all feel discoverers.

I will take your hand and guide you, readers, through a hopefully comfortable walk through a harmonious, tranquil, peaceful, sunny journey, peering at fantastic landscapes of mathematical knowledge and figures. A very long journey, which began many centuries ago and that still has a long way to go. A journey of knowledge, of search for the law that governs the behavior of prime numbers. An adventure trip. Let us enjoy it together. Me, for having written the book, and you, as the reader, joining this group of curious discoverers of those small, simple but very important entities in mathematics, called prime numbers.

Is it worth reading this book? Is it worth spending some time, which is always scarce, to learn things that, at first glance, might seem useless? There is a clean, honest, consistent and grateful world that will not disappoint you. A consistent world of causes and effects. Of promises that are always fulfilled, without debts or mortgages. A calm and friendly world. It is the clean world of mathematics. It is worth taking a look.

I just want to add one more thing. As it is always said in any worthwhile introduction, and without it being rhetorical, all errors that may appear in this book are mine alone. In some cases, I might have kept away from the rigor demanded by mathematics. I have done so for a better understanding. I hope the reader will kindly excuse me, and I hope you have such a good time as I had while I was writing it. Thank you.

José Luis Pérez

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