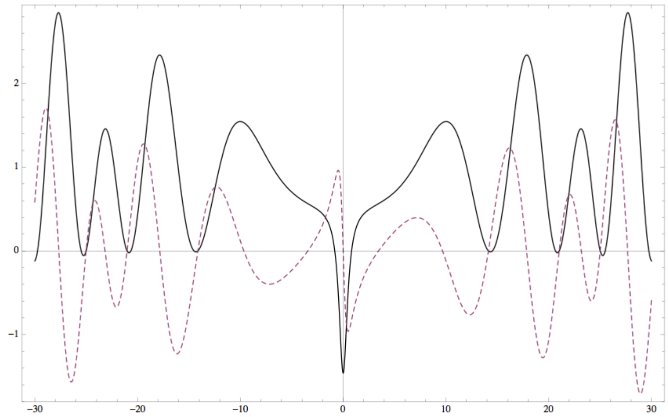

16. The magical analytical formula of $\pi(x)$

Only the infinitely many non-trivial zeros of the Riemann function $\zeta(s)$
know where the infinitely many prime numbers are.

José Luis Pérez (Barcelona, 1957)
(This time, if I may, the quote is mine)

1. **In this chapter we will get to know the analytic formula** found by Riemann for $\pi(x)$ in his 1859 paper. This was the end of his work. The end, yes, although there were many unanswered questions, as we shall see in the remainder of this book.

I would like to advance to the reader that in the last part of his paper, Riemann used truly complicated mathematics. Much more than what we have seen so far, knowing also that I have omitted many developments because of their complexity, replacing them with rather literary presentations. Therefore, and to make things understandable, I shall have to skip many steps and present finished conclusions, that the reader must have faith in.

2. **Adapting $J(x)$.** The first thing to do is to go back to the function $J(x)$, and make a subtle change in it. Figure 16-1 shows it once again, and I draw it from $x = 0$ until $x = 35$. Note that at the points where the function has a step, I have also drawn a mark, that falls exactly in the middle of that step.

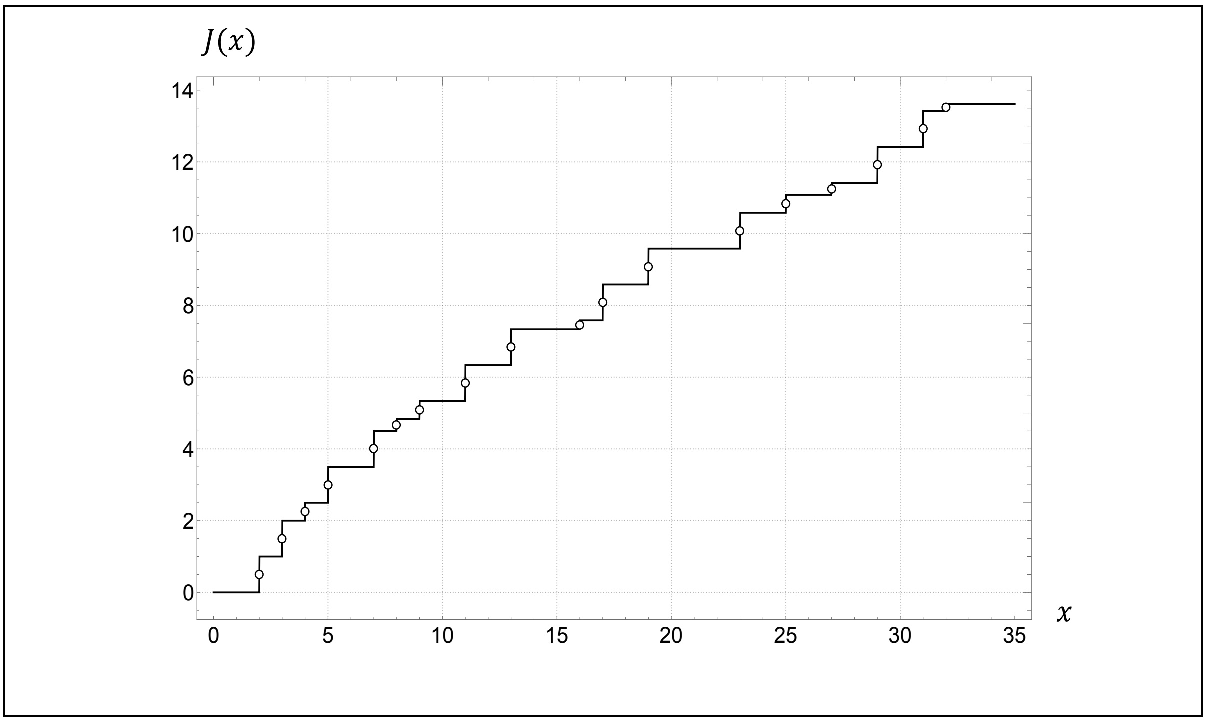

Figure 16-1

What do I mean with this? Well, that the function $J(x)$ will be equal, in each step, to the average of its previous value and subsequent value. For example, $J(10.99) = 5.33333\ldots$, and $J(11.01) = 6.33333\ldots$, so, just on the step, I make $J(11) = 5.83333$ ... And why do I do this? Because it is necessary if we want the mathematics that we will see from now on to work. It is a small change that does not alter the nature of $J(x)$.

How to write this small change mathematically? Well, going back to the original definition of $J(x)$, viewed in the expression 15-1 of the previous chapter based on counting primes and weighted prime powers. As follows:

$$J(x) = \frac{1}{2}\left(\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\leq x}\frac{1}{n}\right)$$

In this expression it is evident that, to the left of the step, primes and prime powers are counted up to $p^n < x$, while, on the right, they are counted until $p^n \leq x$. The sum of these two amounts, divided by two, gives us the value of $J(x)$ right in the step. Later I will discuss why this small change has been necessary.

3. The analytical formula for $J(x)$. Let us start with two important results seen in the last chapter, involving the function $\xi(s)$:

$$\xi(s) = \pi^{-s/2}(s-1)\Gamma(s/2+1)\zeta(s) \quad \text{and} \quad \xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$$

I can get logarithms in both expressions, and I get the following two:

$$\log\xi(s) = -\frac{s}{2}\log\pi + \log(s-1) + \log\Gamma(s/2+1) + \log\zeta(s)$$

$$\log \xi(s) = \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right)$$

where the logarithm of products has become sums of logarithms.

Now, I equal the two expressions of $\log \xi(s)$:

$$-\frac{s}{2}\log \pi + \log(s-1) + \log \Gamma(s/2+1) + \log \zeta(s)$$

$$= \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right)$$

And I solve for $\log \zeta(s)$, which is what interests us:

$$\log \zeta(s) = \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) + \frac{s}{2}\log \pi - \log\,(s-1)$$

$$-\log \Gamma(s/2+1)$$

What does this expression tell us? Well, that $\log \zeta(s)$ depends on five terms. The first is a constant that has value $\log \xi(0) = \log(1/2) = -0.693147$, to 6 decimals. The second term is curious: it involves, by means of an infinite sum, all non-trivial zeros of $\zeta(s)$. We will see that this has important consequences. The last three terms each have a different dependence of s, which is clear, because we already know the functions they depend on.

Now is the time to return to the expression for $\log \zeta(s)$, seen in the expression 15-6 of the previous chapter:

$$\frac{\log \zeta(s)}{s} = \int_{x=0}^{\infty} J(x) \cdot x^{-s-1}\, dx$$

And now comes the moment of truth: Riemann, using mathematical resources that fall outside the scope of this book (Fourier analysis), managed *to solve for* the function $J(x)$ in the above expression, and he wrote it in terms of an integral in which $\log\zeta(s)$ appeared. When I say *to solve for*, I put this in italics, of course, because it is not just solving for an unknown x, but a complicated operation, with many derivatives and integrals of complex variable. Riemann, symbolically, managed to do the following:

$$J(x) = \text{function}\left(\int \log\zeta(s)\right)$$

And as we have just seen that $\log\zeta(s)$ consists of five terms, it also happens that $J(x)$ will be composed of many others. One of them, however, the term that depends on $(s/2)\log\pi$, disappears in the mathematical developments, leaving only four.

The following is the solution reached by Riemann, after much development, for $J(x)$, this being the main result of his paper:

$$\boldsymbol{J(x) = Li(x) - \sum_{\mathrm{Im}[\rho]>0}\left[Li(x^{\rho}) + Li(x^{1-\rho})\right] + \int_{t=x}^{\infty}\frac{dt}{t(t^2-1)\log t} + \log\xi(0)}$$

Expression 16-1