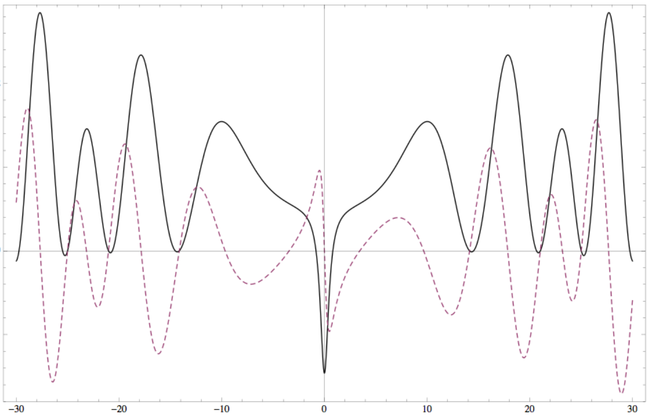

8. The Prime Number Theorem

Music is the pleasure experienced by our mind when counting, without realizing that it is counting.

Gottfried Leibniz (1646-1716), German mathematician.

1. We saw in Chapter 3 that the frequency with which prime numbers appear among natural numbers decreases as we climb in our observation. Let us recall: between 1 and 100 there are 25 primes. Between 1 and 1,000 there are 168. What about between 1 and 10,000? Well, there are exactly 1,229 prime numbers. And what about between 1 and 100,000? The answer is that there are 9,592 primes. At first glance we can see that, approximately, 25% of the numbers between 1 and 100 are primes. Between 1 and 1000 the percentage is 17%, between 1 and 10,000 it is 12.3%, and we find that 9.6% of the numbers between 1 and 100,000 are primes. It is obvious that the number of primes decreases as we seek further.

Moreover, we also saw in Chapter 3, because Euclid proved it, that there are infinitely many prime numbers. Euler also proved it, as we also saw in Chapter 6, in a much more elaborate way using the harmonic series and the Euler product. Primes are endless. There are always more primes. But, if there are infinitely many primes, how are they distributed? We have seen that, at first there are many, but as we go up, there seems to be less and less. Is this always the case? And if it is always like this, does that distribution follow any law? Are we able to know what that law is?

Let us look at the following Table 8.1:

Table 8.1 – Number of primes as a function of *N*

		N	Primes up to N	Ratio
Ten	10^1	10	4	2.5000
One hundred	10^2	100	25	4.0000
One thousand	10^3	1000	168	5.9524
Ten thousand	10^4	10,000	1,229	8.1367
One hundred thousand	10^5	100,000	9,592	10.4254
One million	10^6	1,000,000	78,498	12.7392
Ten million	10^7	10,000,000	664,579	15.0471
One hundred million	10^8	100,000,000	5,761,455	17.3567
One billion	10^9	1,000,000,000	50,847,534	19.6666
Ten billion	10^{10}	10,000,000,000	455,052,511	21.9755
One hundred billion	10^{11}	100,000,000,000	4,118,054,813	24.2833
One trillion	10^{12}	1,000,000,000,000	37,607,912,018	26.5901
Ten trillion	10^{13}	10,000,000,000,000	346,065,536,839	28.8960
One hundred trillion	10^{14}	100,000,000,000,000	3,204,941,750,802	31.2020
One quadrillion	10^{15}	1,000,000,000,000,000	29,844,570,422,669	33.5069
Ten quadrillion	10^{16}	10,000,000,000,000,000	279,238,341,033,925	35.8120
One hundred quadrillion	10^{17}	100,000,000,000,000,000	2,623,557,157,654,233	38.1160
One quintillion	10^{18}	1,000,000,000,000,000,000	24,739,954,287,740,860	40.4204

I have written, in table form, the number of primes there are up to a certain number N, starting with 10, followed by 100, 1,000, and so on. I also added a column to the right, called *Ratio*, which is the result of dividing the number N by the number of primes there are up to N. For example, if $N = 10{,}000$, and up to N there are 1,229 prime numbers, the result of the division is $10{,}000/1{,}229 = 8.1367$. What does this column tell us? It gives us the proportion, or ratio, of the prime numbers there are from 2 to N. For example, up to 10,000, about one out of eight numbers is a prime. Up to one million, it is one out of thirteen. Up to one billion, it is one out of 20. Up to one quintillion, there is one prime number out of every 40. The proportion of primes decreases. And it will continue to decrease, it seems, although it will never reach zero, because we know that there are infinitely many prime numbers. Well, we will stop here for now.

2. The counting function of prime numbers $\pi(x)$. In Chapter 5 we saw what a function was, and I gave several examples. A function is a rule that assigns a set of data, or values, that we call domain, to another set of results, that we will call image. We saw that a function can be defined in several ways: as a table, a mathematical expression, or as a description thereof. Functions can have a proper name, e.g. *speed* function, as an expression of distance traveled and time spent. Or, in general, we can work with abstract functions, which are not associated to a particular magnitude, which we will call as we want: f, g, z, u, etc.

Let us define a very important function that we will use continuously from now on. It is the function that associates the number of primes there are up to a given amount. That is, the function that gives us back, for each input number, the number of primes that there are up to that amount. Unluckily, this function was once given an unfortunate name. It was the mathematician Edmund Landau, a professor at Göttingen, who in 1909 wrote a famous book: *Handbuch der lehre von der Verteilung der Primzahlen*[(1)] (Manual on the distribution of prime numbers). Here, he called the counting function of prime numbers as π. Wow, what bad luck. What a way to confuse things. Because, for everyone, π is the known value 3.141592 ... which is simply the ratio of the length of the circumference to its diameter. However, this name has become usual for this function, and we have to get used to it. Consider that, in developments, the number π will often appear, and depending on the context we can distinguish when π must be interpreted as the famous number, and when as the function $\pi(x)$, that shall be read as follows: *pi* of x. How is the function $\pi(x)$ defined? As follows:

$$\pi(x) = \begin{cases} 0, & if\ x < 2 \\ Number\ of\ primes\ up\ to\ x\ included, & if\ x \geq 2 \end{cases}$$

Here are some values taken by the function $\pi(x)$ depending on the value x: $\pi(2) = 1$, $\pi(3) = 2$, $\pi(4) = 2$, $\pi(5) = 3$, $\pi(6) = 3$, $\pi(7) = 4$, $\pi\ (8) = 4$, $\pi(9) = 4$, $\pi(10) = 4$, $\pi\ (11) = 5$. The function $\pi(x)$ increases its value by one every time we come to a prime number. For example, $\pi(30) = 10$, and $\pi(31) = 11$. But also $\pi(32) = 11$, and we will not see that the function $\pi(x)$ is incremented by one until we reach 37, which is the next prime number, so $\pi(37) = 12$.

$\pi(1) = 0$, as we saw in Chapter 3 that, today, mathematicians do not consider that 1 should be considered a prime. It meets the rules to be a prime, of course, but its obviousness, and the fact that it often spoils theorems and conclusions led mathematicians, already at the end of the nineteenth century, not to include it as a prime. Almost all theorems on prime numbers begin as follows: "Let p be a prime number greater than or equal to 2, ...", so it is not worth dragging 1 as a prime, as it makes no contribution whatsoever.

Can the function $\pi(x)$ be drawn? Of course, it can. And, also, it gives a very beautiful graph. It is shown in figure 8-1 in all its grandeur. I have marked with a dotted line the prime numbers between 0 and 50. In each prime number the function $\pi(x)$ leaps one unit. We see that the